DOI: 10.2478/auom-2019-0003
An. Şt. Univ. Ovidius Constanţa **Vol. 27(1),2019, 49–59**

On the generators of a generalized numerical semigroup

Carmelo Cisto, Gioia Failla and Rosanna Utano

Abstract

We give a characterization on the sets $A \subseteq \mathbb{N}^d$ such that the monoid generated by A is a generalized numerical semigroup (GNS) in $\mathbb{N}^d$. Furthermore we give a procedure to compute the hole set $\mathbb{N}^d \setminus S$, where S is a GNS, if a finite set of generators of S is known.

1 Introduction

Let $\mathbb{N}$ be the set of non negative integers. A *numerical semigroup* is a submonoid S of $\mathbb{N}$ such that $\mathbb{N} \setminus S$ is a finite set. The elements of $H(S) = \mathbb{N} \setminus S$ are called the *holes* of S (or *gaps*) and the largest element in $H(S)$ is known as the *Frobenius number* of S, denoted by $F(S)$. The number $g = |H(S)|$ is named the *genus* of S. It has been proved that every numerical semigroup S has a unique minimal set of generators $G(S)$, that is in S every element is a linear combination of elements in $G(S)$ with coefficients in $\mathbb{N}$. Furthermore the set of minimal generators of a numerical semigroup is characterized by the following: the set $\{a_1, a_2, \ldots, a_n\}$ generates a numerical semigroup if and only if the greatest common divisor of the elements $a_1, a_2, \ldots, a_n$ is 1. For the background on this subject, a very good reference is [9].
In [3] it is provided a straightforward generalization of numerical semigroups in $\mathbb{N}$ for submonoids of $\mathbb{N}^d$: a monoid $S \subseteq \mathbb{N}^d$ is called a generalized numerical semigroup (GNS) if $H(S) = \mathbb{N}^d \setminus S$, the set of holes of S, is a finite set. Also

Key Words: Generalized numerical semigroup, minimal generators.
2010 Mathematics Subject Classification: Primary 20M14, 05A15; Secondary 05A16, 11D07.
Received: 27.01.2018
Accepted: 27.04.2018

in this case the cardinality of $\mathbb{N}^d \setminus S$ is called the *genus* of S. In [3] the tree of generalized numerical semigroups is efficiently calculated up to a given genus and asymptotic properties of the number of generalized numerical semigroups of a given genus are discussed. In this paper we want to extend some ideas and results for classical numerical semigroups to generalized numerical semigroups. We study basic properties of a GNS in order to characterize its minimal system of generators. More precisely, in Section 2 we prove first that every GNS in $\mathbb{N}^d$ has a unique minimal system of generators. Then we investigate under which conditions a finite set $A \subseteq \mathbb{N}^d$ generates a GNS. In Section 3, by using a connection between submonoids of $\mathbb{N}^d$ and power series expansions of rational functions, we deduce an algorithm to compute the set of holes of a GNS, if a finite set of generators of S is given.

2 Minimal generators

Throughout the paper we denote by $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$ the standard basis vectors in $\mathbb{R}^d$ (that is, for $i = 1, \dots, d$, $\mathbf{e}_i$ is the vector whose i-th component is 1 and the other components are zero). Furthermore, if $A \subseteq \mathbb{N}^d$, we denote $\langle A \rangle = \{\lambda_1 \mathbf{a}_1 + \cdots + \lambda_n \mathbf{a}_n \mid \lambda_1, \dots, \lambda_n \in \mathbb{N}, \mathbf{a}_1, \dots, \mathbf{a}_n \in A\}$, that is the submonoid of $\mathbb{N}^d$ generated by the set A. Moreover if $\mathbf{t} \in \mathbb{N}^d$, its i-th component is denoted by $t^{(i)}$.

Lemma 2.1. [9, *Lemma* 2.3] *Let S be a submonoid of $\mathbb{N}^d$. Then $S^* \setminus (S^* + S^*)$ is a system of generators for S. Moreover, every system of generators of S contains $S^* \setminus (S^* + S^*)$.*

Lemma 2.2. *Let S be a GNS of genus g with $H(S) = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \dots, \boldsymbol{h}_{g-1}, \boldsymbol{h}\}$. Let $\boldsymbol{h}$ be a maximal element in $H(S)$ with respect to the natural partial order in $\mathbb{N}^d$. Then $S' = S \cup \{\boldsymbol{h}\}$ is a GNS, in particular $H(S') = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \dots, \boldsymbol{h}_{g-1}\}$ and S' has genus $g - 1$.*

Proof. Let $S' = \langle S \cup \{\mathbf{h}\} \rangle$. S' is a GNS since $S \subseteq S' = \langle S \cup \{\mathbf{h}\} \rangle$, in particular $H(S) \supsetneq H(S')$. Let us prove that S' has genus $g - 1$. We suppose there exists $\mathbf{h}_j \in H(S)$, $j \in \{1, \dots, g-1\}$, such that $\mathbf{h}_j \in S' = \langle S \cup \{\mathbf{h}\} \rangle$. Then $\mathbf{h}_j = \sum_k \mu_k \mathbf{g}_k + \lambda \mathbf{h}$, with $\mathbf{g}_k \in S$. If $\lambda = 0$ then $\mathbf{h}_j \in S$, contradiction. If $\lambda \neq 0$ then $\mathbf{h}_j \geq \mathbf{h}$ against the maximality of $\mathbf{h}$ in $H(S)$. So $\mathbf{h}_j \notin S'$ for $j \in \{1, \dots, g-1\}$, hence $H(S') = \{\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_{g-1}\}$. □

Proposition 2.3. *Every GNS admits a finite system of generators.*

Proof. Let $S \subseteq \mathbb{N}^d$ be a GNS. We prove the statement by induction on the genus g of S. If $g = 0$ then $S = \mathbb{N}^d$, that is generated by the standard basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d\}$. Let $S \subseteq \mathbb{N}^d$ be a GNS of genus $g + 1$ and let $\mathbf{h}$ be a maximal element in $H(S)$ with respect to the natural partial order in $\mathbb{N}^d$. By Lemma 2.2 $S' = S \cup \{\mathbf{h}\}$ is a GNS in $\mathbb{N}^d$ of genus g, that is finitely generated by induction hypothesis. Hence let $G(S')$ be a finite system of generators for S'. We have $\mathbf{h} \in G(S')$ because $\mathbf{h}$ cannot belong to S. So $G(S') \subset S \cup \{\mathbf{h}\}$ and we can denote $G(S') = \{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s, \mathbf{h}\}$ with $\mathbf{g}_i \in S$ for every $i = 1, 2, \ldots, s$. Let $\mathcal{B} = \{\mathbf{g}_1, \ldots, \mathbf{g}_s, \mathbf{h} + \mathbf{g}_1, \mathbf{h} + \mathbf{g}_2, \ldots, \mathbf{h} + \mathbf{g}_s, 2\mathbf{h}, 3\mathbf{h}\}$. By the maximality of $\mathbf{h}$ in $H(S)$ we have $\mathcal{B} \subset S$ and furthermore it is easy to prove that $\mathcal{B}$ is a system of generators for S. Hence S is finitely generated. $\square$

Corollary 2.4. *Every GNS admits a unique finite system of minimal generators.*

Proof. By Lemma 2.1 every GNS admits a unique system of minimal generators, that is $S^* \setminus (S^* + S^*)$, which is contained in every system of generators. By Proposition 2.3 such a system of generators is finite. $\square$

Definition 2.5. Let $\mathbf{t} \in \mathbb{N}^d$, we define the set $\pi(\mathbf{t}) = \{\mathbf{n} \in \mathbb{N}^d \mid \mathbf{n} \leq \mathbf{t}\}$ where $\leq$ is the natural partial order defined in $\mathbb{N}^d$.

Remark 2.6. Notice that for every $\mathbf{t} \in \mathbb{N}^d$ the set $\pi(\mathbf{t})$ is finite and it represents the set of integer points of the hyper-rectangle whose vertices are $\mathbf{t}$, its projections on the coordinate planes, the origin of axes, and the points in the coordinate axes $(t^{(1)}, 0, \ldots, 0), (0, t^{(2)}, 0, \ldots, 0), \ldots, (0, \ldots, 0, t^{(d)})$. If $\mathbf{s} \notin \pi(\mathbf{t})$ then $\mathbf{s}$ has at least one component larger than the respective of $\mathbf{t}$.

Lemma 2.7. *Let $S \subseteq \mathbb{N}^d$ be a monoid. Then S is a GNS if and only if there exists $\boldsymbol{t} \in \mathbb{N}^d$ such that for all elements $\boldsymbol{s} \notin \pi(\boldsymbol{t})$ then $\boldsymbol{s} \in S$.*

Proof. Let S be a GNS in $\mathbb{N}^d$ whose hole set is $H(S) = \{\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_g\}$. Let $t^{(i)} \in \mathbb{N}$ be the largest number appearing in the i-th coordinate of elements in $H(S)$ for $i \in \{1, \ldots, d\}$, in other words $t^{(i)} = \max\{h_1^{(i)}, h_2^{(i)}, \ldots, h_g^{(i)}\}$. It is easy to see that $\mathbf{t} = (t^{(1)}, t^{(2)}, \ldots, t^{(d)}) \in \mathbb{N}^d$ fulfils the thesis.
Conversely, let $\mathbf{t} \in \mathbb{N}^d$ be an element such that for every $\mathbf{s} \notin \pi(\mathbf{t})$ it is $\mathbf{s} \in S$. Therefore if $\mathbf{h} \in \mathbb{N}^d \setminus S$ then $\mathbf{h} \in \pi(\mathbf{t})$, that is $(\mathbb{N}^d \setminus S) \subseteq \pi(\mathbf{t})$ and since $\pi(\mathbf{t})$ is a finite set then S is a GNS. $\square$

For the proof of the next theorem, that is the main result of this paper, we consider that the Frobenius Number of $\mathbb{N}$ (the trivial numerical semigroup) is 0, although it is usually defined to be -1 in the existing literature.

Theorem 2.8. *Let $d \geq 2$ and let $S = \langle A \rangle$ be the monoid generated by a set $A \subseteq \mathbb{N}^d$. Then S is a GNS if and only if the set A fulfils each one of the following conditions:*

1. *For all $j = 1, 2, \ldots, d$ there exist $a_1^{(j)}\boldsymbol{e}_j, a_2^{(j)}\boldsymbol{e}_j, \ldots, a_{r_j}^{(j)}\boldsymbol{e}_j \in A$, $r_j \in \mathbb{N} \setminus \{0\}$, such that $GCD(a_1^{(j)}, a_2^{(j)}, \ldots, a_{r_j}^{(j)}) = 1$ (that is, the elements $a_i^{(j)}$, $1 \leq i \leq r_j$, generate a numerical semigroup).*

2. *For every i, k, $1 \leq i < k \leq d$ there exist $\boldsymbol{x}_{ik}, \boldsymbol{x}_{ki} \in A$ such that $\boldsymbol{x}_{ik} = \boldsymbol{e}_i + n_i^{(k)}\boldsymbol{e}_k$ and $\boldsymbol{x}_{ki} = \boldsymbol{e}_k + n_k^{(i)}\boldsymbol{e}_i$ with $n_i^{(k)}, n_k^{(i)} \in \mathbb{N}$.*

Proof. $\Rightarrow$) If A does not satisfy the first condition for some j then there exist infinite elements $a\mathbf{e}_j$, $a \in \mathbb{N} \setminus \{0\}$, which do not belong to S. If A does not satisfy the second condition for some $i \neq j$, then there are infinite elements $\mathbf{e}_i + n\mathbf{e}_k$ with $n \in \mathbb{N} \setminus \{0\}$ which do not belongs to S.
$\Leftarrow$) For every $j = 1, 2, \ldots, d$, let S_j be the numerical semigroup generated by $\{a_1^{(j)}, a_2^{(j)}, \ldots, a_{r_j}^{(j)}\}$. We denote with $F^{(j)}$ the Frobenius number of S_j. It is easy to verify that for all $n \in \mathbb{N} \setminus \{0\}$, the element $(F^{(j)} + n)\mathbf{e}_j \in \mathbb{N}^d$ belong to S. Let $\mathbf{v} = (v^{(1)}, v^{(2)}, \ldots, v^{(d)}) \in \mathbb{N}^d$ be the element defined by

$$v^{(j)} = \sum_{\substack{i=1\\ i\neq j}}^{d} F^{(i)} n_i^{(j)} + F^{(j)}$$

for any $j = 1, 2, \ldots, d$. Let us prove that $\mathbf{x} \in S$ for all $\mathbf{x} \notin \pi(\mathbf{v})$ so, by Lemma 2.7, S is a GNS.
Let $\mathbf{x} = (x^{(1)}, x^{(2)}, \ldots, x^{(d)}) \in \mathbb{N}^d$ such that $x^{(j)} > v^{(j)}$ for some $j \in \{1, \ldots, d\}$. Then there exists $m_j \in \mathbb{N} \setminus \{0\}$ such that $x^{(j)} = v^{(j)} + m_j$.
If $k_1, k_2, \ldots, k_r \in \{1, 2, \ldots, d\} \setminus \{j\}$ are such that $x^{(k_i)} \leq F^{(k_i)}$ for every $i \in \{1, 2, \ldots, r\}$, so $x^{(k_i)} n_{k_i}^{(j)} \leq F^{(k_i)} n_{k_i}^{(j)}$ for every $i = 1, \ldots, r$, then for every i there exists $p_i \in \mathbb{N}$ such that $F^{(k_i)} n_{k_i}^{(j)} = x^{(k_i)} n_{k_i}^{(j)} + p_i$.
Moreover let $h_1, \ldots, h_s \in \{1, \ldots, d\} \setminus \{j\}$ be the components of $\mathbf{x}$ such that $x^{(h_i)} > F^{(h_i)}$ for every $i \in \{1, \ldots, s\}$, hence $x^{(h_i)}\mathbf{e}_{h_i} \in S$, for all i.

Then we consider the following equalities:

$$\begin{aligned}
\mathbf{x} &= \textstyle\sum_{i=1}^{d} x^{(i)}\mathbf{e}_i = \sum_{i=1}^{r} x^{(k_i)}\mathbf{e}_{k_i} + \sum_{i=1}^{s} x^{(h_i)}\mathbf{e}_{h_i} + x^{(j)}\mathbf{e}_j \\
&= \textstyle\sum_{i=1}^{r} x^{(k_i)}\mathbf{e}_{k_i} + \sum_{i=1}^{s} x^{(h_i)}\mathbf{e}_{h_i} + \left(\sum_{i\neq j}^{d} F^{(i)}n_i^{(j)} + F^{(j)} + m_j\right)\mathbf{e}_j \\
&= \textstyle\sum_{i=1}^{r} \left(x^{(k_i)}\mathbf{e}_{k_i} + F^{(k_i)}n_{k_i}^{(j)}\mathbf{e}_j\right) + \sum_{i=1}^{s} x^{(h_i)}\mathbf{e}_{h_i} + \\
&\quad \textstyle\left(\sum_{i=1}^{s} F^{(h_i)}n_{h_i}^{(j)} + F^{(j)} + mj\right)\mathbf{e}_j \\
&= \textstyle\sum_{i=1}^{r} \left(x^{(k_i)}\mathbf{e}_{k_i} + (x^{(k_i)}n_{k_i}^{(j)} + p_i)\mathbf{e}_j\right) + \sum_{i=1}^{s} x^{(h_i)}\mathbf{e}_{h_i} + \\
&\quad \textstyle\left(\sum_{i=1}^{s} F^{(h_i)}n_{h_i}^{(j)} + F^{(j)} + m_j\right)\mathbf{e}_j \\
&= \textstyle\sum_{i=1}^{r} x^{(k_i)}\left(\mathbf{e}_{k_i} + n_{k_i}^{(j)}\mathbf{e}_j\right) + \\
&\quad \textstyle\sum_{i=1}^{s} x^{(h_i)}\mathbf{e}_{h_i} + \left(\sum_{i=1}^{s} F^{(h_i)}n_{h_i}^{(j)} + \sum_{i=1}^{r} p_i + F^{(j)} + m_j\right)\mathbf{e}_j.
\end{aligned}$$

Therefore $\mathbf{x}$ is a sum of elements in S (note that the first sum is a linear combination of elements in A, whose coefficients are non negative integers). So S is a GNS. □

Corollary 2.9. *Let $S \subseteq \mathbb{N}^d$ be a GNS and let A be a finite system of generators of S. With the notation of the previous theorem for the elements in A, let S_j be the numerical semigroup generated by $\{a_1^{(j)}, a_2^{(j)}, \ldots, a_{r_j}^{(j)}\}$ and $F^{(j)}$ the Frobenius number of S_j, for $j = 1, \ldots, d$. Let $\boldsymbol{v} = (v^{(1)}, v^{(2)}, \ldots, v^{(d)}) \in \mathbb{N}^d$ defined by:*

$$v^{(j)} = \sum_{i\neq j}^{d} F^{(i)}n_i^{(j)} + F^{(j)}.$$

Then $H(S) \subseteq \pi(\boldsymbol{v})$.

Proof. It easily follows from the proof of Theorem 2.8. □

Example 2.10. Let $S \subseteq \mathbb{N}^4$ be the GNS generated by $A = \{(1,0,0,0), (1,0,0,1), (0,1,0,0), (0,1,0,1), (0,0,1,0), (0,0,2,1), (0,0,0,2), (0,0,1,3), (0,0,0,5)\}$. Actually S is a GNS and its hole set is $H(S) = \{(0,0,0,1), (0,0,0,3), (0,0,1,1)\}$. Let us verify that the conditions of Theorem 2.8 are satisfied. The generators described in condition 1) of the previous theorem are $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,2), (0,0,0,5)\}$. About the condition 2) we have to verify that A contains at least one element of the following shapes:

$(n_2^{(1)}, 1, 0, 0)$, $(1, n_1^{(2)}, 0, 0)$, $(1, 0, n_1^{(3)}, 0)$, $(n_3^{(1)}, 0, 1, 0)$,
$(1, 0, 0, n_1^{(4)})$, $(n_4^{(1)}, 0, 0, 1)$, $(0, 1, n_2^{(3)}, 0)$, $(0, n_3^{(2)}, 1, 0)$,
$(0, 1, 0, n_2^{(4)})$, $(0, n_4^{(2)}, 0, 1)$, $(0, 0, 1, n_3^{(4)})$, $(0, 0, n_4^{(3)}, 1)$.

The generators described in condition 2) of the previous theorem are $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (1,0,0,1), (0,1,0,1), (0,0,2,1)\}$. Observe that the set $A' = A \setminus \{(0,0,1,3)\}$ is a set of generators of a GNS S', different from S, with a greater number of holes.

Example 2.11. Let $S \subseteq \mathbb{N}^2$ be the GNS whose hole set is $H(S) = \{(1,0), (2,0), (2,1)\}$. The set of minimal generators of S is $\{(0,1), (1,1), (3,0), (4,0), (5,0)\}$. We can identify $F^{(1)} = 2$, $F^{(2)} = 0$, $n_2^{(1)} = 0$, $n_1^{(2)} = 1$ so $\mathbf{v} = (F^{(2)} n_2^{(1)} + F^{(1)}, F^{(1)} n_1^{(2)} + F^{(2)}) = (2,2)$. In Figure 1 the point $\mathbf{v}$ is marked in red, the couples of nonnegative integers in the red area represent the elements in $\pi(\mathbf{v})$.

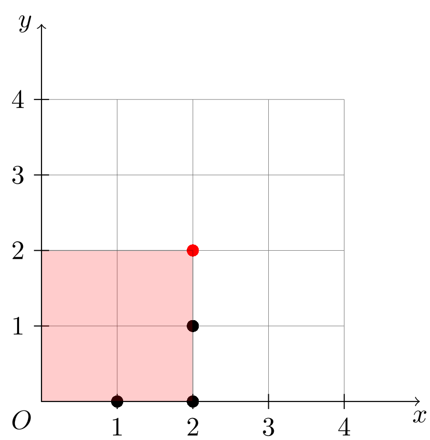

Figure 1:

The holes of S are marked in black and we can see that they are all in the red area, that is $\pi(\mathbf{v})$. Moreover all the points overside the red area are in S. Indeed $\mathbf{v}' = (2,1)$ satisfies Lemma 2.7 too and $|\pi(\mathbf{v}')| < |\pi(\mathbf{v})|$. Anyway this fact does not always occur, as we will see in the next example.

Example 2.12. Let $S \subseteq \mathbb{N}^2$ be the monoid generated by $G(S) = \{(2,0), (0,2), (3,0), (0,3), (1,4), (4,1)\}$.
By Theorem 2.8 S is a GNS. Actually the hole set of S is $H(S) = \{(0,1), (1,0), (1,1), (1,2), (1,3), (1,5), (2,1), (3,1), (5,1)\}$. We have $F^{(1)} = 1$, $F^{(2)} = 1$,

$n_1^{(2)} = 4$, $n_2^{(1)} = 4$, so we consider $\mathbf{v} = (F^{(2)}n_2^{(1)}+F^{(1)}, F^{(1)}n_1^{(2)}+F^{(2)}) = (5,5)$. The set $H(S)$ is contained in $\pi(\mathbf{v})$:

In this case we can argue that it does not exist an element $\mathbf{w} \in \mathbb{N}^2$ such that $\pi(\mathbf{w})$ contains every hole of S and $|\pi(\mathbf{w})| < |\pi(\mathbf{v})|$.

Remark 2.13. Let $S = \langle A \rangle$ be a monoid generated by $A \subseteq \mathbb{N}^d$. For every $j = 1, 2, \dots, n$, we denote with $A_j \subseteq \mathbb{N}^{d-1}$ the set of the elements in $\mathbb{N}^{d-1}$, obtained from the elements in A removing the j-th component. Then the condition 2) of Theorem 2.8 is equivalent to the following statement: for every $j = 1, 2, \dots, d$, $\langle A_j \rangle = \mathbb{N}^{d-1}$.

3 Linear combinations in $\mathbb{N}^d$ with coefficients in $\mathbb{N}$

Let $S \subseteq \mathbb{N}^d$ be a finitely generated monoid and $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ be a system of generators for S. We denote by M the $d \times n$ matrix whose i-th column is the vector $\mathbf{a}_i \in \mathbb{N}^d$ for $i = 1, \dots, n$. It is easy to see that an element $\mathbf{b} \in S$ if and only if the system $M\mathbf{x} = \mathbf{b}$ admits solutions in $\mathbb{N}^n$. In fact this statement is equivalent to say that $\mathbf{b}$ is a linear combination of $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\} \subseteq \mathbb{N}^d$ with nonnegative integer coefficients.

Definition 3.1. Let $A \subseteq \mathbb{N}^d$ be a finite set. We define the polynomial:

$$F_A = \sum_{\mathbf{v} \in A} x^{\mathbf{v}},$$

where $x^{\mathbf{v}} = x_1^{v^{(1)}} x_2^{v^{(2)}} \cdots x_d^{v^{(d)}}$ is the monomial in $K[X_1, \dots, X_d]$ associated to $\mathbf{v} = (v^{(1)}, v^{(2)}, \dots, v^{(d)})$. We consider the power series expansion of $1/(1-F_A)$ the following formal series:

$$P(F_A) = \sum_{k=0}^{\infty} (F_A)^k.$$

The following lemma ([5, Lemma 2.2] for $d = 1$) is obtained by applying *Leibnitz's rule*:

$$(a_1 + a_2 + \cdots + a_m)^n = \sum_{h_1+h_2+\cdots+h_m=n} \frac{n!}{h_1!h_2!\dots h_m!} a_1^{h_1} a_2^{h_2} \cdots a_m^{h_m}.$$

Lemma 3.2. *Let $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n\} \subseteq \mathbb{N}^d$ and $\boldsymbol{b} \in \mathbb{N}^d$. Then $\boldsymbol{b}$ is a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n$ with nonnegative integer coefficients if and only if the coefficient of $x^{\boldsymbol{b}}$ in $P(F_A)$ is nonzero.*

Proof. By Leibnitz's rule we obtain:

$$(F_A)^t = (x_1^{a_1^{(1)}} x_2^{a_1^{(2)}} \cdots x_d^{a_1^{(d)}} + x_1^{a_2^{(1)}} x_2^{a_2^{(2)}} \cdots x_d^{a_2^{(d)}} + \cdots + x_1^{a_n^{(1)}} x_2^{a_n^{(2)}} \cdots x_d^{a_n^{(d)}})^t =$$

$$= \sum K \cdot x_1^{a_1^{(1)}h_1 + a_2^{(1)}h_2 + \cdots a_n^{(1)}h_n} \cdot x_2^{a_1^{(2)}h_1 + a_2^{(2)}h_2 + \cdots a_n^{(2)}h_n} \cdot \cdots \cdot x_d^{a_1^{(d)}h_1 + a_2^{(d)}h_2 + \cdots a_n^{(d)}h_n},$$

where the sum is extended to $h_1, \ldots, h_n \in \mathbb{N}$ with $h_1 + \cdots + h_n = t$ and K is a nonzero coefficient.

If $\mathbf{b} = \sum_{i=1}^n \lambda_i \mathbf{a}_i$, set $t = \sum_{i=1}^n \lambda_i$, then $x^{\mathbf{b}}$ is a monomial in $(F_A)^t$. Conversely, if $x^{\mathbf{b}}$ has nonzero coefficient in $P(F_A)$ then

$$x^{\mathbf{b}} = x_1^{a_1^{(1)}h_1 + a_2^{(1)}h_2 + \cdots a_n^{(1)}h_n} \cdot x_2^{a_1^{(2)}h_1 + a_2^{(2)}h_2 + \cdots a_n^{(2)}h_n} \cdot \cdots \cdot x_d^{a_1^{(d)}h_1 + a_2^{(d)}h_2 + \cdots a_n^{(d)}h_n}$$

with $h_i \in \mathbb{N}$ for $i = 1, \ldots, n$ that is $\mathbf{b} = \sum_{i=1}^n h_i \mathbf{a}_i$.

$\square$

Definition 3.3. Let $A = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} \subseteq \mathbb{N}^d$ with $\mathbf{a}_i = (a_i^{(1)}, a_i^{(2)}, \ldots, a_i^{(d)})$ for $i = 1, 2, \ldots, n$, and let $\mathbf{b} \in \mathbb{N}^d$.
Let $t = \min\{\sum_{j=1}^d a_i^{(j)} \mid i = 1, 2, \ldots, n\}$. We define the positive integer

$$N_{\mathbf{b}} := \left\lfloor \frac{\sum_{j=1}^d b^{(j)}}{t} \right\rfloor.$$

Proposition 3.4. *Let $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\} \subseteq \mathbb{N}^d$ and $\boldsymbol{b} \in \mathbb{N}^d$. Then $\boldsymbol{b} \in \langle A \rangle$ if and only if the coefficient of $x^{\boldsymbol{b}}$ is nonzero in the polynomial:*

$$F(x_1, x_2, \ldots, x_d) = \sum_{k=0}^{N_b} (F_A)^k.$$

Proof. By lemma 3.2 it is enough to show that the coefficient of $x^{\mathbf{b}}$ is zero in $F(x_1, \ldots, x_d)$ if and only if it is zero also in $P(F_A)$, that is $\sum_{k=0}^{\infty} (F_A)^k$.
We suppose that the coefficient of $x^{\mathbf{b}}$ is nonzero in $P(F_A)$. Then there exists $r \in \mathbb{N}$ such that $x^{\mathbf{b}}$ is a monomial in $(F_A)^r$. By Leibnitz's rule we obtain:

$$(F_A)^r = (x_1^{a_1^{(1)}} x_2^{a_1^{(2)}} \cdots x_d^{a_1^{(d)}} + x_1^{a_2^{(1)}} x_2^{a_2^{(2)}} \cdots x_d^{a_2^{(d)}} + \cdots + x_1^{a_n^{(1)}} x_2^{a_n^{(2)}} \cdots x_d^{a_n^{(d)}})^r$$

$$= \sum_{\mathbf{h}} K \cdot x_1^{a_1^{(1)}h_1 + a_2^{(1)}h_2 + \cdots a_n^{(1)}h_n} \cdot x_2^{a_1^{(2)}h_1 + a_2^{(2)}h_2 + \cdots a_n^{(2)}h_n} \cdot \cdots \cdot x_d^{a_1^{(d)}h_1 + a_2^{(d)}h_2 + \cdots a_n^{(d)}h_n},$$

where $\mathbf{h} = (h_1, \ldots, h_n)$ with $h_1 + h_2 + \cdots + h_n = r$ and K is the correspondent coefficient, but we do not need its exact value.
If $x_1^{b^{(1)}} x_2^{b^{(2)}} \ldots x_d^{b^{(d)}}$ appears in the sum, then there exist $h_1, h_2, \ldots, h_n$ with $h_1 + h_2 + \cdots + h_n = r$, such that the following equalities are satisfied:

$$\begin{aligned} a_1^{(1)} h_1 + a_2^{(1)} h_2 + \cdots a_n^{(1)} h_n &= b^{(1)} \\ a_1^{(2)} h_1 + a_2^{(2)} h_2 + \cdots a_n^{(2)} h_n &= b^{(2)} \\ &\vdots \\ a_1^{(d)} h_1 + a_2^{(d)} h_2 + \cdots a_n^{(d)} h_n &= b^{(d)}. \end{aligned}$$

We sum the righ-hand side and the left-hand side of all equalities, obtaining that:

$$\begin{aligned} r &= h_1 + h_2 + \cdots + h_n \leq \\ &\leq (a_1^{(1)} + a_1^{(2)} + \cdots + a_1^{(d)}) h_1 + (a_2^{(1)} + a_2^{(2)} + \cdots + a_2^{(d)}) h_2 + \cdots + \\ &+ (a_n^{(1)} + a_n^{(2)} + \cdots + a_n^{(d)}) h_n = b^{(1)} + b^{(2)} + \cdots + b^{(d)}. \end{aligned}$$

Eventually, if $t = \min\{\sum_{j=1}^{d} a_i^{(j)} \mid i = 1, 2, \ldots, n\}$ then $\frac{\sum_{j=1}^{d} a_i^{(j)}}{t} \geq 1$ for $i = 1, 2, \ldots, d$. So we can divide the right-hand side of inequality by t and we obtain:

$$\begin{aligned} r &= h_1 + h_2 + \cdots + h_n \leq \\ &\leq \frac{\sum_{j=1}^{d} a_1^{(j)}}{t} h_1 + \frac{\sum_{j=1}^{d} a_2^{(j)}}{t} h_2 + \cdots + \frac{\sum_{j=1}^{d} a_n^{(j)}}{t} h_n = \frac{b^{(1)} + b^{(2)} + \cdots + b^{(d)}}{t} \end{aligned}$$

It follows that $r \leq N_{\mathbf{b}}$. So, if the coefficient of $x^{\mathbf{b}}$ in $P(F_A)$ is nonzero then the greatest power in which it is obtained is at last $N_{\mathbf{b}}$, for greater powers we are sure that monomial does not appear. □

An application of the previous proposition is the following criterion for the existence of $\mathbb{N}$-solutions in a linear system with nonnegative integer coefficients.

Corollary 3.5. *Let M be a $d \times n$ matrix with entries in $\mathbb{N}$ whose columns are the vectors of the set $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ and let $\boldsymbol{b} \in \mathbb{N}^d$. Then the linear system $M\boldsymbol{x} = \boldsymbol{b}$ admits solutions $\boldsymbol{x} \in \mathbb{N}^n$ if and only if the coefficient of $x^{\boldsymbol{b}}$ is nonzero in the polynomial:*

$$F(x_1, x_2, \ldots, x_d) = \sum_{k=0}^{N_b} (F_A)^k.$$

The previous arguments suggest the following results.

Corollary 3.6. *Let $S \subseteq \mathbb{N}^d$ be a GNS, $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ be a finite system of generators for S and $\boldsymbol{v} \in \mathbb{N}^d$. Then $\boldsymbol{v} \in S$ if and only if the coefficient of $x^{\boldsymbol{v}}$ is nonzero in the polynomial:*

$$F(x_1, x_2, \ldots, x_d) = \sum_{k=0}^{N_v} (F_A)^k.$$

If S is a GNS and a finite system of generators for S is known, then Corollary 3.6 provides a way to establish whether an element $\mathbf{v} \in S$. Furthermore it can be done with a finite computation, that is the building of a polynomial.

Remark 3.7. Recall that if $S \subseteq \mathbb{N}^d$ is a GNS and A a finite system of generators for S, by Theorem 2.8 A satisfies the following conditions:

1. For all $j = 1, 2, \ldots, d$, there exist $a_1^{(j)}\mathbf{e}_j, a_2^{(j)}\mathbf{e}_j, \ldots, a_{n_j}^{(j)}\mathbf{e}_j \in A$ such that $GCD(a_1^{(j)}, a_2^{(j)}, \ldots, a_{n_j}^{(j)}) = 1$
2. For every $i, k \in \{1, 2, \ldots, d\}$ with $i < k$ there exist $\mathbf{x}, \mathbf{y} \in A$ such that $\mathbf{x} = \mathbf{e}_i + n_i^{(k)}\mathbf{e}_k$ and $\mathbf{y} = \mathbf{e}_k + n_k^{(i)}\mathbf{e}_i$ with $n_i^{(k)}, n_k^{(i)} \in \mathbb{N}$.

For every $j = 1, 2, \ldots, d$, let S_j be the numerical semigroup generated by $\{a_1^{(j)}, a_2^{(j)}, \ldots, a_{n_j}^{(j)}\}$. We denote by $F^{(j)}$ the Frobenius number of S_j. Let $\mathbf{v} = (v^{(1)}, v^{(2)}, \ldots, v^{(d)}) \in \mathbb{N}^d$ be the element defined by

$$v^{(j)} = \sum_{i \neq j}^{d} F^{(i)} n_i^{(j)} + F^{(j)}.$$

It is proved that $H(S) \subseteq \pi(\mathbf{v})$ (Corollary 2.9), and $\pi(\mathbf{v})$ is a finite set.

We conclude giving a simple algorithm to compute the set of holes of S, that is $H(S)$, if a finite system of generators for S is known.

Algorithm.

Let $S \subseteq \mathbb{N}^d$ be a GNS and $A = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be a finite system of generators of S. To compute $H(S)$ we have to do the following steps:

1. Compute the element $\mathbf{v}$ of the Remark 3.7.
2. For all $\mathbf{x} \in \pi(\mathbf{v})$ we verify: if $\mathbf{x}$ is not a $\mathbb{N}$-linear combination of elements in A then $\mathbf{x} \in H(S)$. This check can be done by Corollary 3.6.

At the end of the second step the set $H(S)$ is computed.

References

[1] Bras-Amorós, M.: Fibonacci-like behavior of the number of numerical semigroups of a given genus. Semigroup Forum **76**, 379–384 (2008).

[2] Bras-Amorós, M.: Bounds on the number of numerical semigroups of a given genus. J. Pure Appl.Algebra **213**, 997–1001 (2009).

[3] Failla, G., Peterson, C., Utano, R.: Algorithms and basic asymptotics for generalized numerical semigroups in $\mathbb{N}^d$, Semigroup Forum **92**(2), 460–473 (2016).

[4] Fröberg, R., Gottlieb, C., Häggkvist, R.: On numerical semigroups. Semigroup Forum **35**, 63–83 (1986/1987).

[5] Martino, I., Martino, L.: On the variety of linear recurrences and numerical semigroups. Semigroup Forum **88**, 569–574 (2014).

[6] Pisón-Casares, P., Vigneron-Tenorio, A.: $\mathbb{N}$-solutions to linear systems over $\mathbb{Z}$. Linear Algebra Its Appl. **384**, 135–154 (2004).

[7] Rosales, J.C.: On finitely generated submonoids of $\mathbb{N}^k$. Semigroup Forum **50**, 251–262 (1995).

[8] J. C. Rosales, P. A. García-Sánchez, On Cohen-Macaulay and Gorenstein simplicial affine semigroups, Proc. Edinburgh Math. Soc. 41 (1998), 517–537.

[9] Rosales, J.C., García-Sánchez, P.A.: Numerical Semigroups, Developments in Mathematics, vol. 20. Springer, New York (2009).

[10] Rosales, J.C., García-Sánchez, P.A., García-García, J.I., Jiménez Madrid, J.A.: The oversemigroups of a numerical semigroup, Semigroup Forum **67**(1), 145–158, (2003).

Carmelo CISTO
Universitá di Messina,
Dipartimento di Scienze Matematiche e Informatiche, Scienze Fisiche e Scienze della Terra,
Viale Ferdinando Stagno D'Alcontres 31,
98166 Messina, Italy
Email: carmelo.cisto@unime.it

Gioia FAILLA,
Universitá Mediterranea di Reggio Calabria,
Dipatimento DIIES,
Via Graziella, Feo di Vito,
Reggio Calabria, Italy.
Email: gioia.failla@unirc.it

Rosanna UTANO
Universitá di Messina,
Dipartimento di Scienze Matematiche e Informatiche, Scienze Fisiche e Scienze della Terra,
Viale Ferdinando Stagno D'Alcontres 31,
98166 Messina, Italy
Email: rosanna.utano@unime.it